Basic Terminologies in Convex Optimization

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This short tutorial will explain some very basics terminologies related to convex optimization. These terminologies should be well understood before we learn convex optimization.

There are five terms that would be explained here which are :

1. Linear combination
2. Affine combination
3. Convex combination
4. Convex area
5. Convex function

We start from a linear combination.

1 Linear combination

Consider a 2D cartesian coordinate with x_1 variable for horizontal axis, and x_2 for vertical axis. Let say that there are two points in this axis namely A(x_{A1}, x_{A2})=(2,2) and B(x_{B1}, x_{B2})=(7,3) as shown in the Fig.1.

It is more convinience to write coordinate point A and point B in form of vector, thus $A = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

Now, we can linearly combine vector A and B.

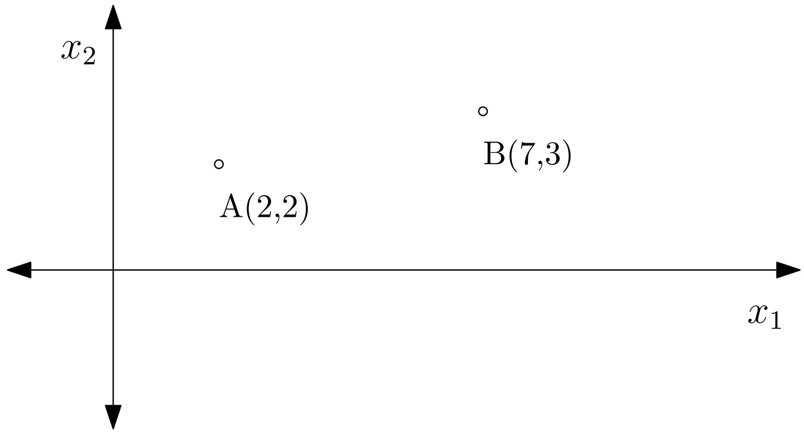

Figure 1: Two points on x_1-x_2 cartesian coordinate.

For example 2A + B will yield another point, say C, which

$$C = 2A + B = 2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}. \tag{1}$$

Another linear combination for example : D = -3A + B, which is

$$D = -3A + B = -3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \tag{2}$$

Point C and D can be plotted as shown in Fig. 2.

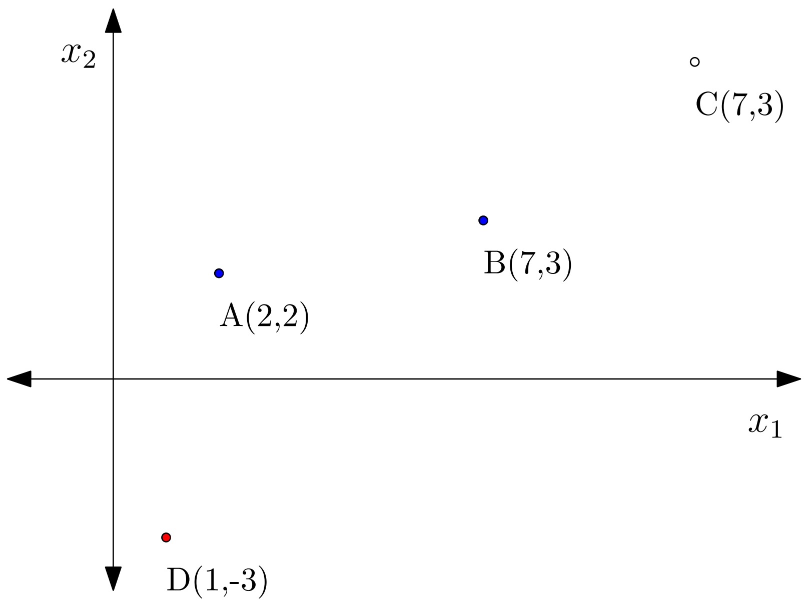

Figure 2: The linear combination of A and B to produce C and D.

By this simple illustration, we can produce any point in $x_1 - x_2$ cartesian coordinate by using linear combination of point A and point B.

We can state this simple statement mathematically as:

Let A and B any points (A≠B) in cartesian coordinate $x_1 - x_2$. Then **any points P** in this coordinate can be expressed as linear combination of A and B. That is

$$P = a_1 \cdot A + a_2 \cdot B \tag{3}$$

for **any real values** a_1 and a_2.

This statement is illustrate in Fig.3.

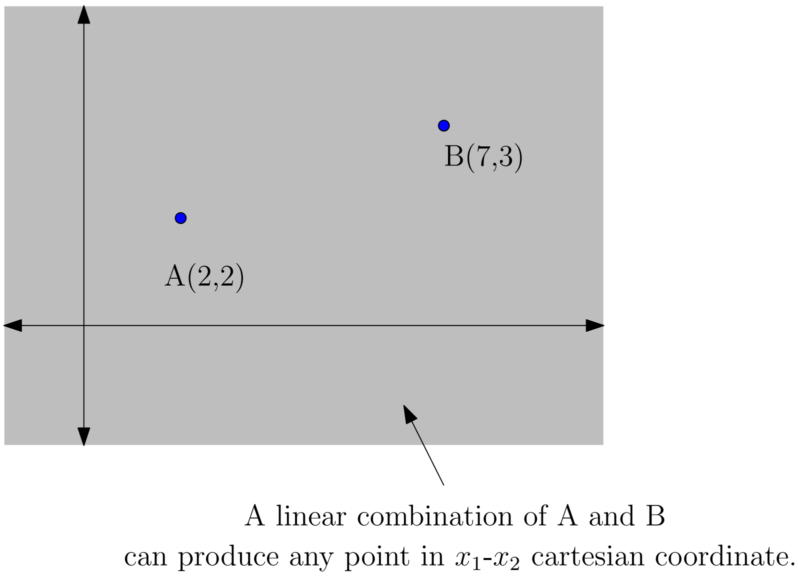

Figure 3: Area of possible results of linear combination of A and B.

2 Affine combination

We consider again Point A(2,2) and point B(6,3) as in previous example. Also, we consider again a linear combination of Point A and Point B as

$$P = a_1 \cdot A + a_2 \cdot B \tag{4}$$

but this time we cannot select a_1 and a_2 freely. Instead, we select that

$$a_1 + a_2 = 1 \tag{5}$$

For example, if $a_1 = 0.5$ and $a_2 = 0.5$, then we have

$$P = 0.5 \cdot A + 0.5 \cdot B = 0.5 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 0.5 \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 2.5 \end{bmatrix} \tag{6}$$

This point is just middle point of A and B.

Again we choose another combination, which is $a_1 = 0.9$ and $a_2 = 0.1$, then we have

$$P = 0.9 \cdot A + 0.1 \cdot B = 0.9 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.1 \end{bmatrix} \tag{7}$$

This point is lie to the line segment AB and close to A (Fig.4)

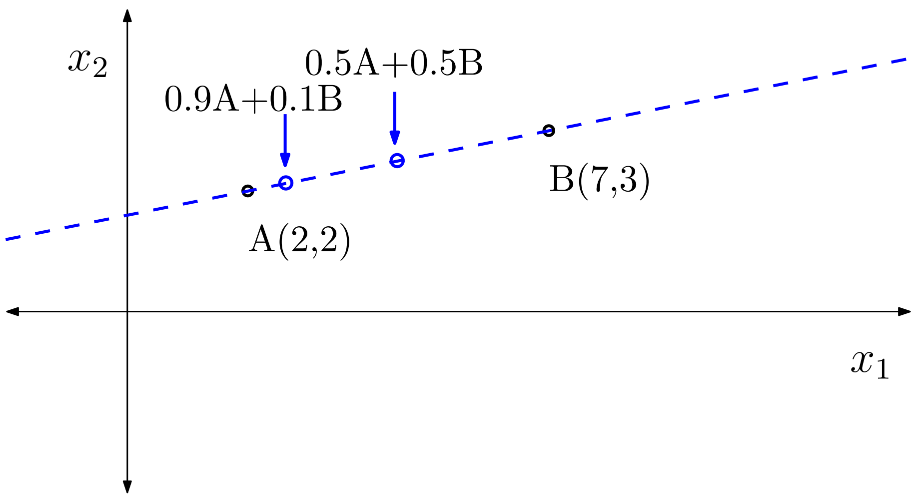

Figure 4: Ilustration of a linear combination of point A and B for $a_1 = 0.5$ $a_2 = 0.5$ and $a_1 = 0.9$ $a_2 = 0.1$

How if we choose $a_1 = 1$ and $a_2 = 0$, in this case we get P = A.

Next how if we choose $a_1 = 1.1$ and $a_2 = -0.1$, that is

$$P = 1.1 \cdot A - 0.1 \cdot B = 1.1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 0.1 \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.9 \end{bmatrix} \tag{8}$$

Now the point P from the last combination of A and B lies at the left

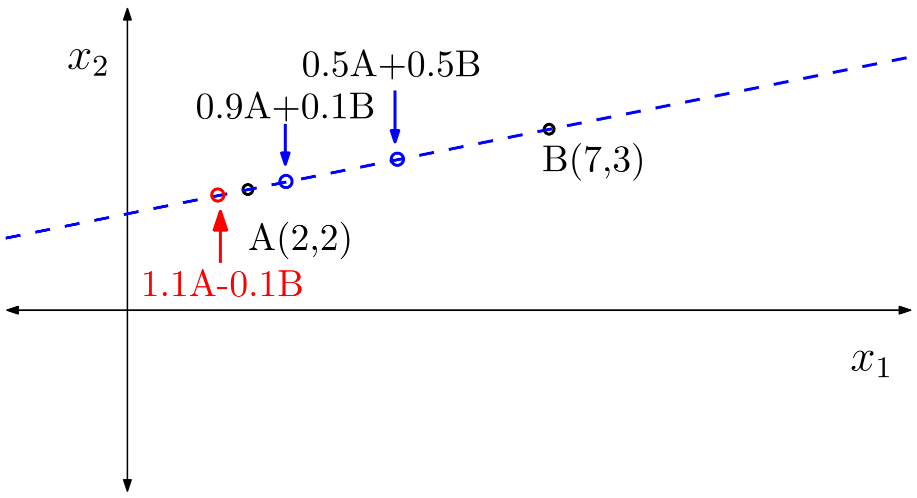

Figure 5: Ilustration of a linear combination of point A and B for $a_1 = 0.5$ $a_2 = 0.5$ and $a_1 = 0.9$ $a_2 = 0.1$

of A (Fig.5).

From this illustration, we conclude this simple fact from linear combination of $P = a_1A + a_2B$ with $a_1 + a_2 = 1$ gives the result of point P that lies on straight line that connects A and B.

The parameter a_1 controls the influence of point A to P. Starting from $a_1 = 0.5$ and increase it to 1 means that the point P shift from middle of segment AB to point A itself. If a_1 is increased so that it greater than 1 (and a_2 less than 0), then the point P now move to the left of A (i.e. outside of the segment AB, but still in the straight line that connect A and B).

From this simple observation, we can further generalize the role of a_2. The parameter a_2 on the other hand controls the point B. Starting from $a_2 = 0.5$ we are at mid point of line segment AB. Increasing a_2 to 1, we move point P from mid point to the point B itself along the line segment AB. If we further increase a_2 greater than 1 (i.e. $a_1 \leq 0$), then the point P move outside of the segment AB at the right side of point B.

We can conclude now that a linear combination of A and B which is $P = a_1A + a_2B$ produce a point P that lies on the **straight line** that connect A and B. This type of linear combination is called AFFINE combination.

AFFINE combination is combination of point A and B (i.e. $P = a_1 A + a_2 B$ with $a_1 + a_2 = 1$) produces a point P that lies on the **straight line** that connects A and B. In other word, AFFINE combination is a special case of linear combination where $a_1 + a_2 = 1$ for any real value a_1 and a_2.

3 Convex combination

In previous section we know that Affine combination is a special case of linear combination where $a_1 + a_2 = 1$ and a_1 and a_2 are real numbers. Now we will consider a special case of Affine combination, which is called **Convex combination**. In affine combination, we can select a_1 any real number and a_2 adjusts correspondingly. In convex combination, we have $a_1 + a_2 = 1$ AND a_1 and a_2 are **positives** real number.

Convex combination is a special case of affine combination, where a_1 and a_2 should be positives.

In complete words: Convex combination of point A and B is $P = a_1 A + a_2 B$ where $a_1 + a_2 = 1$ and $a_1 \geq 0$ and $a_2 \geq 0$.

What is physical interpretation of convex combination?

Recall again affine combination in previous section. If $a_1 = 0.5$ and $a_2 = 0.5$ then we know P is at the mid-point of line segment AB. If $a_1 = 0.9$ and $a_2 = 0.1$ then P is at line segment AB and closed to point A. If $a_1 = 1$ and $a_2 = 0$ then we are at point A. If $a_1 = 0.1$ and $a_2 = 0.9$ then point P is at line segment AB and closed to point B. If $a_2 = 1$ and $a_1 = 0$, then point P is point B itself.

Fig.**??**

From this short discussion, we are quite clear now with the terminology of linear combination, affine combination and convex combination.

Using this basis, we are ready to discuss the definition of convex area and convex function in the next sections.

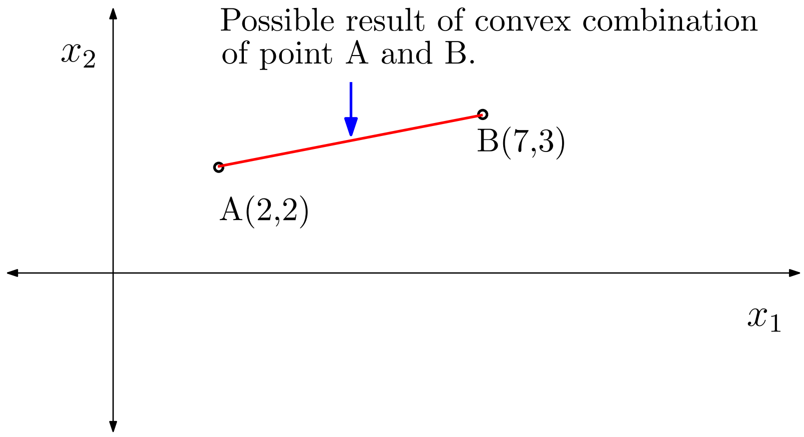

Figure 6: The possible result of Convex combination of point A and B is the line segment AB itself.

4 Convex area

We consider three shapes as shown in Fig.7.

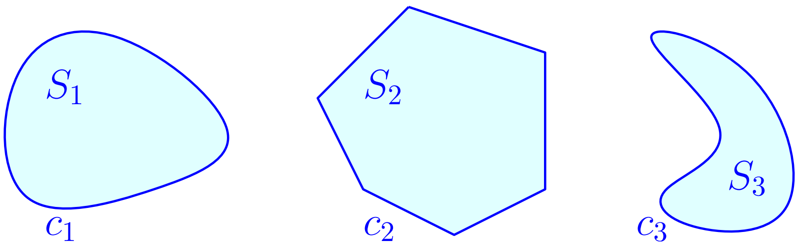

Figure 7: Example of three areas

Lets call first area be S_1 and bounded by curve c_1, second area be S_2 bounded by c_3 and the last curve S_3 and bounded by c_3.

If we take any two points in S_1, says A and B, than any convex combination of A and B will produce point a P which is also in S_1 (Fig.8).

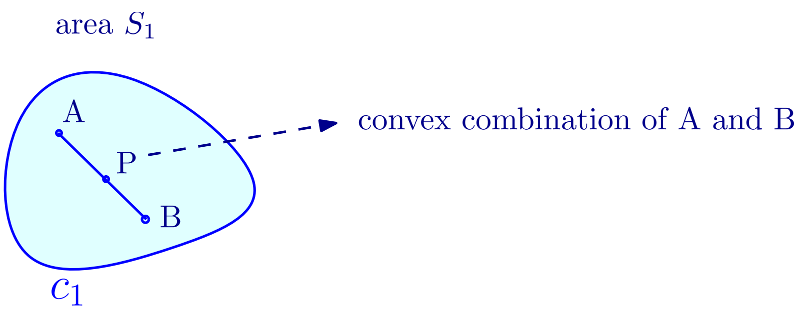

Figure 8: Convex combination of any two points in S_1 is also in S_1

Area S_1 is called a convex area.

With similar reasoning, we also know that area S_2 (which is a polytope) is also a convex area.

Now we consider area S_3. S_3 is not a convex area, since we can select point A and B as shown in Fig.9, and some convex combinations of A and B do not lie in S_3. Therefore S_3 is not a convex area.

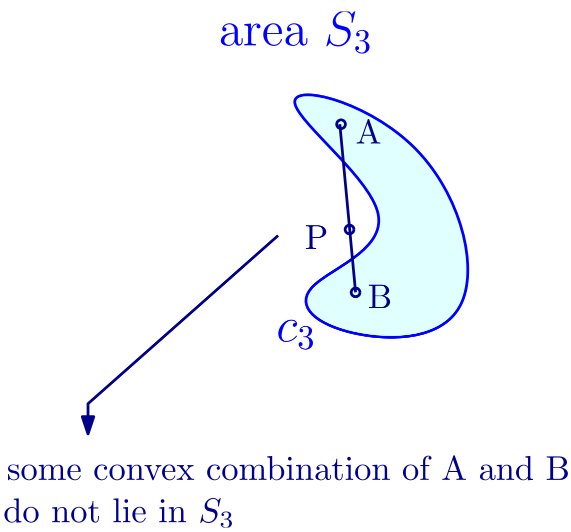

Figure 9: Ilustration of area S_3.

From this illustration, we now can state a convex area as

> An area S is called convex, if for any convex combination of two points in S is also in S.
>
> **Mathematically**: S is convex if $A \in S$ and $B \in S$ then point P which is $P = a_1 A + a_2 B$ for $a_1 + a_2 = 1$ and $a_1 \geq 0$ and $a_2 \geq 0$ is also $\in S$.

5 Convex function

In convex optimization, we often encounter the terminology of *convex function*. Consider two functions $f(x)$ and $g(x)$ as shown in Fig.10.

First, let us discuss the plot of *f(x)* as shown in Fig. 10.

Let x_A and x_B any two distinct points in x-axis. The values of function at these points are $f(x_A)$ and $f(x_B)$ respectively.

Now let x_P be any convex combination of x_A and x_B, i.e. $x_P =$

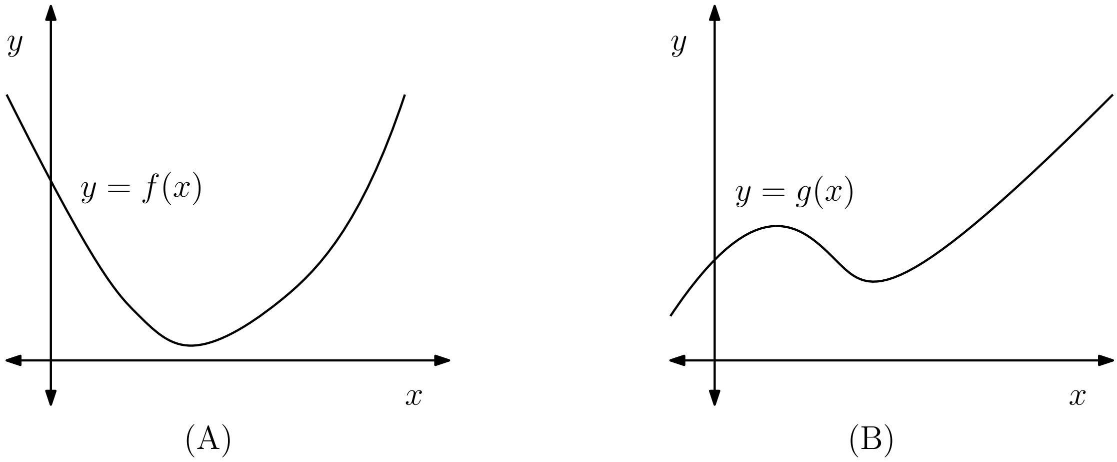

Figure 10: Two function of different nature f(x) and g(x)

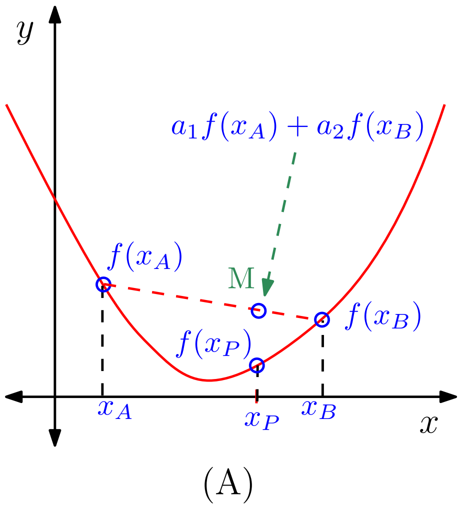

Figure 11: Function *f(x)* in detail.

$a_1 x_A + a_2 x_B$, for any non-negative a_1 and a_2 and $a_1 + a_2 = 1$. In other word

$$f(x_P) = f(a_1 x_A + a_2 x_B) \tag{9}$$

The value of *f()* at x_P is $f(x_A)$ as shown in the Fig.11.

Now we do a similar convex combination on $f(x_A)$ and $f(x_b)$ using a_1 and a_2 that we have used for x_A and x_B to produce x_P. That is

$$M = a_1 f(x_A) + a_2 f(x_B) \tag{10}$$

M is the result of this convex combination.

Now from Fig.11, it is very clear that $M \geq f(x_P)$. In other word,

$$a_1\, f(x_A) + a_2\, f(x_A) \geq f(a_1\, x_A + a_2\, x_B) \tag{11}$$

Eq. 11 just simply want to say that $M \geq f(x_P)$.

If for any x_A and x_B and $a_1 \geq 0$ and $a_2 \geq 0$ and $a_1 + a_2 = 1$ and the Equation 11 holds, then the function $f(x)$ is said to be convex.

We can summarize this result into following note:

A function f(x) is said to be convex if

$$a_1 f(x_A) + a_2 f(x_B) \geq f(a_1 x_A + a_2 x_B)$$

for any x_A and x_B ($x_A \neq x_B$) and for any $a_1 \geq 0$ and $a_2 \geq 0$ and $a_1 + a_2 = 1$.

The reader can easily understand the physical interpretation on convex function as shown in Fig.10.

Now, it is Reader's task to prove that function $g(x)$ as shown in Fig.11.(B) is not a convex function.

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