Basic Concepts of Primal-Dual in Linear Programming

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This tutorial is written in order to understand basically what is primal and dual concept in Linear Programming. Understanding this concept will greatly help us in understanding a more advanced optimization concept such as interior point method algorithm. Before we

1 A Simple Example

Consider the following situation.

Let say that there is two kinds of food (**Food A** and **Food B**) made from **fish** and **flour**. A portion of Food **A** need **1 kg fish** and **1 kg flour**. A portion of Food **B** need **1/2 kg** fish and **1 kg flour**. One portion of food **A** if sold will gives benefit of **50 cents**, while one portion of food **B** if sold will gives benefit of **30 cents**.

If there are **10 kg of fish** and **12 kg of flour**.

How many portion of food **A** and **B** should be made in order to maximize benefit?

Ans

Well, as we see this is a common **Linear Programming** (LP) problem. This technique has been normally given in Senior High School Math.

Now let us back on our problem. Let us say that we produce x_1 number of portion of food **A**, and x_2 number of portion of food **B**.

This total production should be constrained by the amount of fish and flour available.

Thus, constraint for the fish :

$$1 \cdot x_1 + 0.5 \cdot x_2 \le 10 \tag{1}$$

and constraint for the flour is

$$1 \cdot x_1 + 1 \cdot x_2 \le 12 \tag{2}$$

Objective function is

$$Z = 50 \cdot x_1 + 30 \cdot x_2 \tag{3}$$

If we only produce 1 portion of food A and 1 portion of food B, then the benefit we get will be $50 \cdot 1 + 30 \cdot 1 = 80$. And we only use 1+0.5 = 1.5kg fish, and 1+1 =2kg flour.

Of course, since producing food A gives more benefit, it is logical to produce more food A than food B.

But we hold that for the moment. Now let solve this LP using simplex algorithm.

First we introduce slack variables s_1 and s_2 into our constraints : So

$$x_1 + 0.5 \cdot x_2 \le 10 \tag{4}$$

becomes

$$x_1 + 0.5 \cdot x_2 + s_1 = 10 \tag{5}$$

And

$$x_1 + x_2 \le 12 \tag{6}$$

changes to

$$x_1 + x_2 + s_2 = 12 \tag{7}$$

And rewrite the objective function as

$$Z - 50 \cdot x_1 - 30 \cdot x_2 = 0 \tag{8}$$

Collecting 5, 7 and 8 we obtain the complete equation for simplex algorithm:

$$x_1 + 0.5x_2 + s_1 = 10 \tag{9}$$

$$x_1 + x_2 + s_2 = 12 \tag{10}$$

$$Z - 50 \cdot x_1 - 30 \cdot x_2 = 0 \tag{11}$$

Now creating the simplex table:

Table #1

	x_1	x_2	s_1	s_2	RHS
	1	0.5	1	0	10
	1	1	0	1	12
Z	-50	-30	0	0	0

RHS is right hand size terms.

Before we start with simplex algorithm, it is worth to observe the plot of constraints as depicted in Fig. 1.
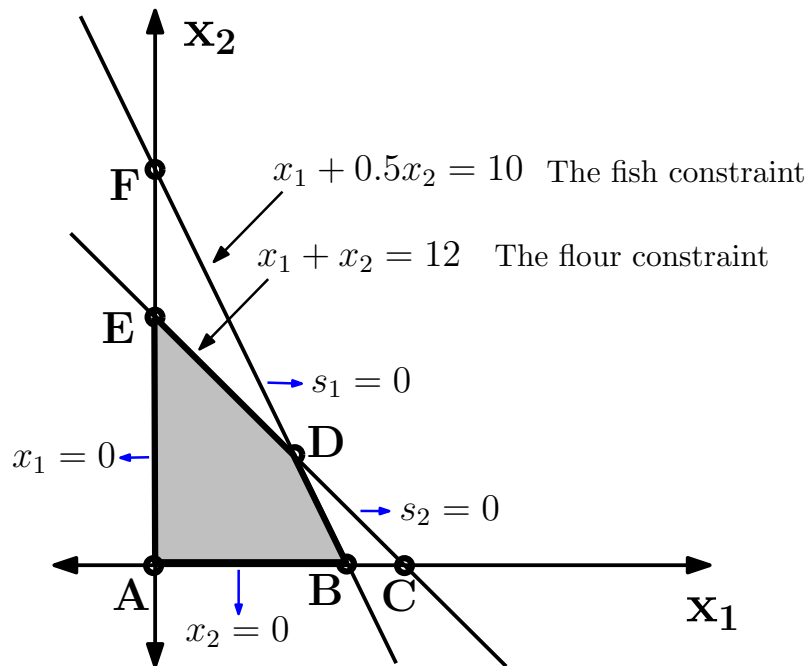

Figure 1: The plot of contraints

In Fig. 1, that the constraints form a **feasible region** ABDEA.

The area are bounded with lines $x_1 = 0$, $x_2 = 0$, $x_1 + x_2 = 12$, $0.5x_1 + x_2 = 10$.

It is interesting to note that the line $x_1 + 0.5x_2 = 12$ is also called line $s_1 = 0$, since the constraint with slack variable is $x_1 + x_2 + s_1 = 12$, then $x_1 + x_2 = 12$ is achieved when $s_1 = 0$.

With similar reasoning, It is interesting to note that the line $x_1 + x_2 = 12$ is also called line $s_1 = 0$, since the constraint with slack variable is $x_1 + x_2 + s_1 = 12$, then $x_1 + x_2 = 12$ is achieved when $s_1 = 0$.

The solution of LP programming normally lies on either A or B or D or E points, i.e. at vertex of feasible area OR at line segments of AB or BD or DE or EA. So either on VERTEX or on LINE SEGMENTS. The Simplex Algorithm works by exploiting this fact. It start at A, and then determine whether to go to B or to E, and so on (checking each points at vertex of the feasible area).

Using this fact, let us now start with Simplex Algorithm.

We start from point **A** $((x_1, x_2) = (0,0))$. Next, where should we go? A or E?

Before we answer this, we observe that there are 6 columns in this simplex table. Second column is x_1 column, and third column is x_2 column.

In Simplex Algorithm, first we should determine, which variable is critical, x_1 or x_2. Since producing x_1 gives benefit 50 (indicated by -50 in x_1 column), and x_2 which only give benefit 30 (as indicated by -30 in x_2 column), therefore we select x_1 column, and this correspond to the action that we will go into x_1 direction. In other word, we will go to either B or C.

If we go to B(10,0) then our objective function Z = 50x10+30x0 = 500. If we go to C(12,0), then our objective function Z = 50x12+30x0

= 600.

It seem that we should go to point C. But from Fig. 1, we observe that point C is not in feasible region. We can only go to point B. So how we know this in simplex?

Using x_1 column as reference, we know that we can use the first row to find B, which is $10/1 = 10$ (that is : RHS divide by coef_of_x_1), and second row to find C which is ($12/1=12$). We collect this information as a new column (INDIK column). We know that we should choose the smallest value. Selecting x_1 column, and first row, we obtain a pivot element which is coloured blue in the following table.

Table #2.

	x_1	x_2	s_1	s_2	RHS	INDIK
	1	0.5	1	0	10	10
	1	1	0	1	12	12
Z	-50	-30	0	0	0	

Now we perform elementary row operation using pivot element to eliminate 1 in second row and -50 on the third row, so that in x_1 column, only pivot element is not zero.

```
second row = second row - first row
```
and
```
third row = third row + 50 x first row
```

We obtain (INDIK column is removed again)

Table #3

	x_1	x_2	s_1	s_2	RHS
	1	0.5	1	0	10
	0	0.5	-1	1	2
Z	0	-5	50	0	500

Oke great. In this Table #3, we see that now we are at point B. And at this point (10,0) our objective function has value of 500 (the most right below element).

Since column x_1 has been chosen, the only **x** variable left is x_2, therefore we choose this column. Since the first row has been chosen, then there is no more choice left except the second row. The element at x_2 column on the second row is 0.5. So this is our next pivot (we color it blue).

Table #4

	x_1	x_2	s_1	s_2	RHS
	1	0.5	1	0	10
	0	**0.5**	-1	1	2
Z	0	-5	50	0	500

Normalize this pivot element to 1 by multiplying second row with 2, we obtain

Table #5

	x_1	x_2	s_1	s_2	RHS
	1	0.5	1	0	10
	0	**1**	-2	2	4
Z	0	-5	50	0	500

Again we have to nullize other elements in this column to zero using this pivot element, therefore we perform the elementary row operation

```
first row = first row - 0.5 x second row
and
third row = third row + 5 x second row
```

We obtain

Table #6

	x_1	x_2	s_1	s_2	RHS
	1	0	2	-1	8
	0	**1**	-2	2	4
Z	0	0	40	10	520

OK, our simplex method has reach final stage as shown in Table #6. In this table, we see that objective function gives the highest value 520 at $x_1 = 10$ (right element of the first row), and $x_2 = 4$ (right element of the second row).

Our optimal solution $(x_1, x_2) = (10,4)$. And our optimal benefit is 520.

2 Interpretation of Final Simplex Table : Sensitivity of LP

Table #6 shows the final Simplex Table.

	x_1	x_2	s_1	s_2	RHS
	1	0	2	-1	8
	0	**1**	-2	2	4
Z	0	0	40	10	520

Interesting to observe the score 40 and 10 in the last row. 40 corresponds to s_1 column and s_2 corresponds to s_2 column.

Oke, our situation is in this problem is, we are limited by the total amount of fish to be 10 unit, and amount of flour are limited by the total amount of 12 unit.

What happen if the amount of fish is increased to 11 and amount of flour is kept constant at 12kg. Will our benefit increase by making more food A or B? How much?

To answer this problem, we again solve it using simplex. Our initial table would be

Table #1B

	x_1	x_2	s_1	s_2	RHS
	1	0.5	1	0	11
	1	1	0	1	12
Z	-50	-30	0	0	0

And our final table is

Table #6B

	x_1	x_2	s_1	s_2	RHS
	1	0	2	-1	10
	0	**1**	-2	2	2
Z	0	0	40	10	560

We see here that our advantage increase from 520 to 560 (i.e. by 40), so now we understand 40 in column s_1. This is a profit margin if we increase the amount of fish to 11 instead of 10. In this case also we observe that we increase food A production by 2 and decrease food B

by 2.

Finally, let us consider the last scenario, we keep the amount of fish to 10 and increase flour from 12 to 13. What is the profit?

Again we solve this problem using simplex.

Initial Table :

Table #1C

	x_1	x_2	s_1	s_2	RHS
	1	0.5	1	0	10
	1	1	0	1	13
Z	-50	-30	0	0	0

Final table

Table #6C

	x_1	x_2	s_1	s_2	RHS
	1	0	2	-1	7
	0	1	-2	2	6
Z	0	0	40	10	530

In this final table, we see that the total profit is 530, only increase 10 from 520 in initial problem. Now we get better understanding of original final Table. Put it again here :

Table 6

	x_1	x_2	s_1	s_2	RHS
	1	0	2	-1	8
	0	**1**	-2	2	4
Z	0	0	40	10	520

Column s_1 from top to bottom:

2, -2, 40.

This clearly mean that if we add 1 unit of fish, in order to get new optimum, then the production of A should be increased by 2 and production of B should be decreased 2, and the profit is increased by 40.

Column s_2 from top to bottom:

-1, 2, 10.

This clearly mean that if we add 1 unit of flour, in order to get new

optimum, then the production of A should be decreased by 1 and production of B should be increased by 2, and by doing this, the profit is increased by 10.

This situation has been verified by our examples.

3 Primal-Dual in Linear Programming

In our previous example, we see how simplex method work. Initially we start at Table #1, and finally we arrived at Table #6.

Here we put again Table #1 and Table #6

Table #1

	x_1	x_2	s_1	s_2	RHS
	1	0.5	1	0	10
	1	1	0	1	12
Z	-50	-30	0	0	0

Table #6

	x_1	x_2	s_1	s_2	RHS
	1	0	2	-1	8
	0	**1**	-2	2	4
Z	0	0	40	10	520

Now we are talking about the dual form the previous LP problem. The original LP is
Constraints :

$$x_1 + 0.5 \cdot x_2 \leq 10 \tag{12}$$

$$x_1 + x_2 \leq 12 \tag{13}$$

Objective function is

$$Z = 50 \cdot x_1 + 30 \cdot x_2 \tag{14}$$

The dual of this LP is (change objective to constraint and constraint to objective)

Constraints :

$$y_1 + y_2 \geq 50 \tag{15}$$

$$0.5 \cdot y_1 + y_2 \geq 30 \tag{16}$$

Objective function is to **minimize**

$$Y = 10 \cdot y_1 + 12 \cdot y_2 \tag{17}$$

This new LP is called dual of previous LP. Since this one is called **Dual**, therefore it makes sense that the original one is called **Primal**. Observe that

1. objective of primal becomes constraints of dual

2. constraints of primal becomes objective of dual

3. if primal is to maximize then dual is to minimize

4. if primal has constraint of $\leq$, then dual has constraint of $\geq$

Introducing new variable r_1 and r_2, the modified LP becomes Constraints :

$$y_1 + y_2 + r_1 = 50 \tag{18}$$

$$0.5 \cdot y_1 + y_2 + r_2 = 30 \tag{19}$$

Objective function is to **minimize**

$$Y = 10 \cdot y_1 + 12 \cdot y_2 \tag{20}$$

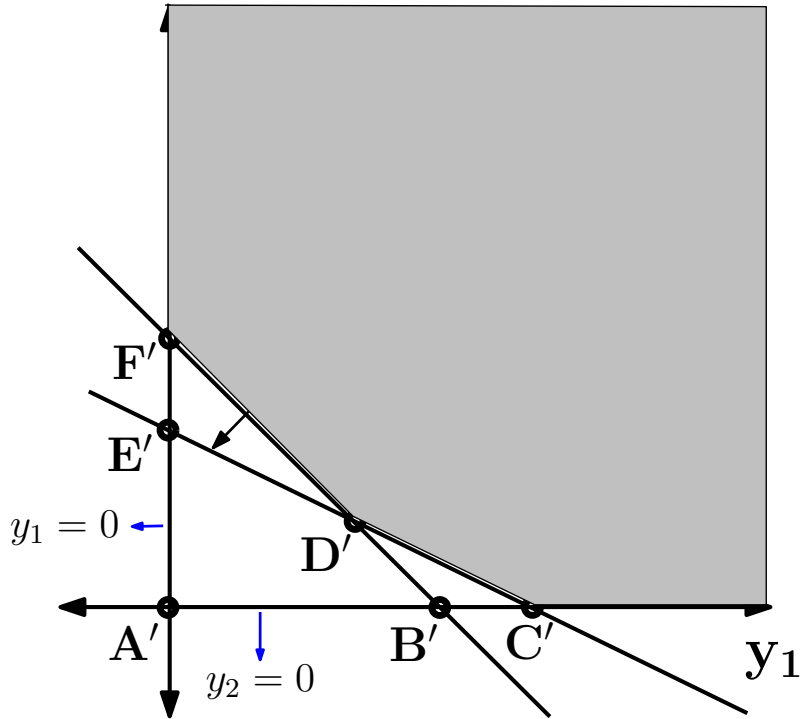The plot of constraints is as shown in Fig. 2
Using simplex, the initial table is

Figure 2: The constraints of dual

Table #1C

	x_1	x_2	s_1	s_2	RHS
	1	0	2	-1	8
	0	**1**	-2	2	4
Z	0	0	40	10	520

Since we are minimizing, please be careful selecting each point, we have to make sure that we are at point A' then go to either F', or C' and so on in feasible region.

Doing this, we arrive in final table as (say that this is as Table #6C).

Table #6C

	y_1	y_2	r_1	r_2	RHS
	0	1	1	-2	10
	1	**0**	-2	2	40
Y	0	0	-8	-4	520

Compare this with final state of original Table, we obtain

Table #6

	x_1	x_2	s_1	s_2	RHS
	1	0	2	-1	8
	0	**1**	-2	2	4
Z	0	0	40	10	520

We see that these two table is very similar. It has the same similar optimal value which is 520. The position of main variable and slack variable in primal and dual are interchanged.

In other word, if we find the optimal value in main variable of the primal, than we know that this is the optimal value for the dual in slack variable, and vice versa.

Even though that physical interpretation of dual in LP is not really obvious, but some at least some aspect has been understood so far.

4 Citation of this work

If you find this work useful, please cite it as follows:

Usman, Koredianto, 2017, "Basic Concepts of Primal-Dual in Linear Programming", Telkom University, available : korediantousman.staff.telkomuniversity.ac.id